notes on Transmon quantum computer

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Contents

0.1 Transmon

0.1.1 LC oscillator

For a LC circuit, the Hamiltonian is:

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \tag{1}$$

$$[\Phi, Q] = i\hbar \tag{2}$$

introducing ladder operators:

$$\begin{aligned} a &= Q' - i\Phi' & a^\dagger &= Q' + i\Phi' \\ Q' &= \tfrac{Q}{\sqrt{2\hbar/Z}} & \Phi' &= \tfrac{\Phi}{\sqrt{2\hbar Z}} \\ H &= \hbar\omega(a^\dagger a + \tfrac{1}{2}) & [a, a^\dagger] &= 1 \\ Z = \omega L &= \tfrac{1}{\omega C} = \sqrt{\tfrac{L}{C}} \end{aligned} \tag{3}$$

0.1.2 the Josephson junction

$$\phi = 2\pi \frac{\Phi}{\Phi_0} \quad \Phi_0 = \frac{h}{2e} \quad \text{flux quantum} \tag{4}$$

$$I = I_c \sin\phi \tag{5}$$

$$E = \int_0^t IV\ dt = \int_0^t I \frac{\Phi_0}{2\pi}\frac{d\phi}{dt}\ dt = \frac{\Phi_0}{2\pi}\int_0^\phi I_c \sin\phi\ d\phi = \underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_J}(1 - \cos\phi) \tag{6}$$

for small flux the Josephson junction behaves just like an inductor:

$$E \approx \frac{\Phi_0 I_c}{2\pi}\frac{\phi^2}{2} = \frac{I_c \pi}{\Phi_0}\Phi^2 \tag{7}$$

0.1.3 Jaynes-Cummings model

$$H = \underbrace{\hbar\omega_c a^\dagger a}_{\text{field}} + \underbrace{\hbar\omega_a \frac{1}{2}\underbrace{(|e\rangle\langle e| - |g\rangle\langle g|)}_{\sigma_z}}_{\text{atom}} + \underbrace{\hbar\Omega\frac{ES}{2}}_{\text{int}} \tag{8}$$

$$E = E_{ZPF}(a + a^\dagger) \quad S = \underbrace{|e\rangle\langle g|}_{\sigma_+} + \underbrace{|g\rangle\langle e|}_{\sigma_-} \tag{9}$$

where ZPF stands for zero-point field. Under rotating wave approximation:

$$H_{\text{int}} = \hbar\Omega\frac{1}{2}(a\sigma_+ + a^\dagger\sigma_-) \tag{10}$$

the possible transition is:

$$|e,n\rangle \leftrightarrow |g,n+1\rangle \tag{11}$$

the Hamiltonian in this basis is:

$$H = \begin{pmatrix} n\hbar\omega_c + \frac{1}{2}\hbar\omega_a & \frac{1}{2}\hbar\Omega\sqrt{n+1} \\ \frac{1}{2}\hbar\Omega\sqrt{n+1} & (n+1)\hbar\omega_c - \frac{1}{2}\hbar\omega_a \end{pmatrix} \tag{12}$$

with eigenvalues:

$$E_\pm = (n+\frac{1}{2})\hbar\omega_c \pm \frac{1}{2}\hbar\underbrace{\sqrt{(\omega_a-\omega_c)^2 + \Omega^2(n+1)}}_{\Omega_n \quad \text{Rabi frequency}} \tag{13}$$

0.2 Controlling cavities

Notes from reading Controlling Error-Correctable Bosonic Qubits by Philip Reinhold (2019) Chapter 2: Controlling cavities

0.2.1 Quantization of EM field

$$\text{Scully (1.1.11)} \qquad a = \frac{m\omega x + ip}{\sqrt{2m\hbar\omega}} \quad a^\dagger = \frac{m\omega x - ip}{\sqrt{2m\hbar\omega}} \tag{14}$$

$$\text{Scully (1.1.27)} \qquad \boldsymbol{E} = \sum_{\boldsymbol{k}} \underbrace{\boldsymbol{e_k}}_{\text{polarization}} \underbrace{E_{\boldsymbol{k}}}_{\sqrt{\hbar\omega_{\boldsymbol{k}}/2\epsilon_0 V}} \hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} + HC \tag{15}$$

where V is the volume of the resonator.

0.2.2 Changing frames

$$\psi_1 = U\psi_0 \tag{16}$$

$$\partial_t\psi_1 = (\partial_t U)\psi_0 + U(\partial_t\psi_0) = (\partial_t U)\psi_0 + U(-iH\psi_0) \tag{17}$$

$$= -i\underbrace{\left[i(\partial_t U)U^\dagger + U(HU^\dagger)\right]}_{\tilde{H}}\psi_1 \tag{18}$$

Remark 1: Philip used $\partial_t\psi_0 = -iH\psi_0$ but had $\partial_t\psi_1 = iH\psi_1$

0.2.3 Direct Control part 1

$$H = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \underbrace{\epsilon(t)\Omega_{\boldsymbol{k}}(a^\dagger_{\boldsymbol{k}} + a_{\boldsymbol{k}})}_{\text{doesn't look like } \boldsymbol{E}} \tag{19}$$

Question 1: the last term doesn't look like $\boldsymbol{E}$

For a single mode: **Question 2:** What is his Ω and why is it gone

$$H = \underbrace{\omega a^\dagger a}_{H_0\text{, detuning term that we will murder}} + \epsilon(t)(a^\dagger + a) \tag{20}$$

0.2.4 Rotating Frame: to kill H_0

$$\boxed{U_i = e^{-iH_i t}} \tag{21}$$

Result: $H = \omega a^\dagger a + f(a^\dagger, a) \longrightarrow \tilde{H} = f(a^\dagger e^{-i\omega t}, a e^{i\omega t})$

0.2.5 Direct Control part 2

$$\tilde{H} = \epsilon(t)(a^\dagger e^{-i\omega t} + a e^{i\omega t}) \tag{22}$$

$$\tilde{U} = e^{-i\int \tilde{H}(t)dt} \equiv \underbrace{e^{\alpha a^\dagger - \alpha^* a}}_{\text{displacement operator } D_\alpha} \tag{23}$$

Conclusion 1: Direct control using EM field can only produce coherent states, which are basically displacements of the ground state in the phase space.

0.2.6 States

$$\text{Coherent state: } |\alpha\rangle = D_\alpha|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{24}$$

$$\text{Fock/number state:} \tag{25}$$

0.2.7 Pauli Matrices

$$\sigma_x\sigma_y = i\sigma_z \quad \sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \tag{26}$$

$$(\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma})^2 = I \longrightarrow e^{i\theta\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}} = I\cos\theta + \hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}\ i\sin\theta \tag{27}$$

$$R_{\hat{\boldsymbol{u}}}(\alpha) := e^{i(-\frac{\alpha}{2})\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}} \tag{28}$$

0.2.8 Resonant Jaynes-Cummings Model (without control)

$$H = \underbrace{\omega a^\dagger a}_{H_0} + \underbrace{\frac{\omega_a}{2}\sigma_z}_{H_1} + \frac{\Omega}{2}(a^\dagger\sigma_- + a\sigma_+) \tag{29}$$

$$\tilde{H} = \frac{\Omega}{2}(a^\dagger\sigma_- e^{-i\Delta)t} + HC) \qquad \Delta := \omega - \omega_a \tag{30}$$

Note: the transmon qubit acts as the *atom* in the textbook Jaynes-Cummings model

Question 3: Near (2.12) "the form of the drive is $H_d = \frac{\Omega(t)}{2}\sigma_x$"

0.2.9 Dispersive Jaynes-Cummings Model (without control)

$$H = H_0 + H_1 + \frac{\chi}{2}a^\dagger a\sigma_z \qquad \chi := \frac{\Omega^2}{2\Delta} \tag{31}$$

$$\tilde{H} = \chi a^\dagger a|e\rangle\langle e| \tag{32}$$

0.2.10 Dispersive Jaynes-Cummings Model (with control)

$$\tilde{H}_{\text{ctrl}} = \chi a^\dagger a |e\rangle\langle e| + [\Omega(t)\sigma_- + \epsilon(t)a + HC] \tag{33}$$

0.2.11 The Toolkit Five

Question 4: By letting $\phi = -\chi t, \Omega = \epsilon = 0$, define **entangling conditional phase** $C_\phi = e^{i\phi\, a^\dagger a|e\rangle\langle e|}$, show $C_\pi = I_c|g\rangle\langle g| + \underbrace{e^{i\pi a^\dagger a}}_{\Pi = P_{\text{even}} - P_{\text{odd}}} |e\rangle\langle e|$. Therefore, $C_\pi = \underbrace{\sum_{k \text{ even}} |k\rangle\langle k|}_{P_{\text{even}}} I_q + \sum_{k \text{ odd}} |k\rangle\langle k|\sigma_z$

Question 5: Show $e^{i\phi a^\dagger a}|\alpha\rangle = |e^{i\phi}\alpha\rangle$

$e^{i\phi a^\dagger a}|\alpha\rangle = e^{i\phi a^\dagger a} D_\alpha |0\rangle = e^{i\phi a^\dagger a} e^{\alpha a^\dagger - \alpha^* a}|0\rangle \overset{???}{=} e^{e^{i\phi}\alpha a^\dagger - (e^{i\phi}\alpha)^* a}|0\rangle$

Question 6: "A narrow drive, centered around zero frequency in this rotating frame, applied to the transmon, will induce Rabi oscillations **if and only if the cavity contains zero photons**, this is a photon number selective qubit drive."

$$R_\phi^{(n)}(\theta) = |n\rangle\langle n| \underbrace{R_\phi(\theta)}_{\textbf{definition ???}} + (I_c - |n\rangle\langle n|) I_q \tag{34}$$

Question 7: In section (2.3.1) before Figure 2.4, **time** is not included in any discussion or equation, why $t \approx 0.53\mu s$ is special and what does it have do with the previous circuit?

0.2.12 Wigner Function

$$W_\alpha(\rho) = \frac{2}{\pi} x \langle D_{-\alpha} \Pi D_\alpha \rangle \tag{35}$$

0.2.13 Bloch Sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \tag{36}$$

Remark 2:

Are you saying that he defines the operator $R_\phi(\theta)$ by $R_\phi(\theta) := \cos\phi\sigma_x + \sin\phi\sigma_y$? If this is a definition, shouldn't θ appear on the RHS ?

Actually if my English understanding is correct, he is saying that the quantity $\cos\phi\sigma_x + \sin\phi\sigma_y$ is **an axis**, therefore a vector, but isn't this quantity an operator/matrix ?

Dzmitry: Yes it is a little bit strange to specify the rotation axis as a matrix, but it is unfortunately common in quantum computing. If you use your equation $R_{\hat{\boldsymbol{u}}}(\alpha) := e^{i(-\frac{\alpha}{2})\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}}$. These notations mean that $\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma} = \cos\phi\,\boldsymbol{\sigma}_x + \sin\phi\,\boldsymbol{\sigma}_y$, and your rotation axis $\hat{\boldsymbol{u}}$ in the Block sphere has components $(\cos\phi, \sin\phi, 0)$. In other words, instead of specifying vector $\hat{\boldsymbol{u}}$ he gives a product $\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}$, which can be used to find vector $\hat{\boldsymbol{u}}$.

Me: Now I understand why she was talking about parameterization with only one parameter ϕ, she defaulted $\theta = \frac{\pi}{2}$. Is it ever necessary to perform a rotation around an axis that is not on the $z = 0$ plane?

Speaking of this, I noticed that the rotation operator $R_{\hat{\boldsymbol{u}}}(\alpha) := e^{i(-\frac{\alpha}{2})\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}}$ is time-independent, unlike unitary operators of the form e^{iHt} whose response speed is limited by energy. A rotation in the Bloch sphere can be completed instantly! Is this why $z = 0$ is sufficient? We only care about the initial and final states but not the rotation trajectory to get there?

0.2.14 Berry/Geometric phase

0.2.15 SNAP: selective number-dependent arbitrary phase

$$S(\boldsymbol{\theta}) = \sum_k e^{i\theta_k}|k\rangle\langle k| \tag{37}$$

$$R_\phi(-\pi) = e^{i(-\frac{-\pi}{2})\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}} = I\ \cos\frac{\pi}{2} + \hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}\ i\sin\frac{\pi}{2} = \hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}\ i = i[\cos\phi\sigma_x + \sin\phi\sigma_y] \tag{38}$$

$$R_0(\pi) = e^{i(-\frac{\pi}{2})\hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}} = I\ \cos\frac{-\pi}{2} + \hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}\ i\sin\frac{-\pi}{2} = \hat{\boldsymbol{u}}\cdot\boldsymbol{\sigma}\ (-i) = -i[\cos 0\sigma_x + \sin 0\sigma_y] = -i\sigma_x \tag{39}$$

$$R_\phi(-\pi)R_0(\pi) = \cos\phi\sigma_x\sigma_x + \sin\phi\sigma_y\sigma_x = \cos\phi I + \sin\phi(-i\sigma_z) = e^{i(-\phi)\sigma_z} \tag{40}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-i\phi)^n(|0\rangle\langle 0| - |1\rangle\langle 1|)^n = \underbrace{e^{-i\phi}}_{\text{berry's phase}}|0\rangle\langle 0| + \underbrace{e^{i\phi}}_{\text{berry's phase}}|1\rangle\langle 1| \tag{41}$$

Therefore,

$$\prod_{k=0}^{N} R_{\phi_k}^{(k)}(-\pi)R_0^{(k)}(\pi) = \sum_{k=0}^{N} e^{-i\phi_k}|k,0\rangle\langle k,0| + e^{i\phi_k}|k,1\rangle\langle k,1| \tag{42}$$

$$= S(-\boldsymbol{\theta})|0\rangle\langle 0| + S(\boldsymbol{\theta})|1\rangle\langle 1| \tag{43}$$

0.2.16 Quantum Optimal Control

$$H = \underbrace{H_0}_{\text{drift Hamiltonian}} + u_j(t) \underbrace{H_j}_{\text{control Hamiltonians}} \tag{44}$$

0.2.17 Quantum Gates

$$H = R_y(\frac{\pi}{2})R_z(\pi) \tag{45}$$

$$\text{QFT: } x_i|i\rangle \to y_i|i\rangle \qquad y_i = \frac{1}{\sqrt{N}} x_n (\omega_N)^{ni} \qquad \omega_N = e^{\frac{2\pi i}{N}} \tag{46}$$

0.2.18 The Lindblad Master equation

$$\text{Closed system: Neumann: } \dot{\rho} = -i[H, \rho] \tag{47}$$

$$\text{Open system: Lindblad: } \dot{\rho} = -i[H, \rho] + [V_n \rho V_n^\dagger - \frac{1}{2}\{\rho, V_n^\dagger V_n\}] \tag{48}$$